Flocking and reorientation transition in the 4-state active Potts model

UNIVERSITÄT DES SAARLANDES ZBP

Matthieu Mangeat[1], Swarnajit Chatterjee[2,1], Raja Paul[2] and Heiko Rieger[1]

[1]Center for Biophysics & Department for Theoretical Physics, Saarland University, Saarbrücken, Germany

[2]School of Mathematical & Computational Sciences, Indian Association for the Cultivation of Science, Kolkata, India

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Introduction

Collective motions are widely observed in nature (bird flocks, fish schools, bacteria swarms) and also studied in various artificial systems (liquid droplets, rolling colloids, vibrated polar disks). They consist of a large number of active particles which consume energy to self-propel and interact via alignment and/or repulsion. For large densities and low noise spontaneous synchronized motion of large clusters of particles, denoted as flocks, emerges. This flocking transition is an out-of-equilibrium phenomenon, leading to several theoretical models.

The first theoretical model displaying a flocking transition was the Vicsek model (VM) [1, 2]. The active particles perform a ballistic motion with constant velocity and align with the local average particle direction with the rule $\theta_i(t+dt) = \langle\theta(t)\rangle_r + \eta\xi_i(t)$. This alignment is similar to a ferromagnetic interaction where η plays the role of the temperature. A collective motion emerges at low temperatures and high densities due to a spontaneous breaking of the continuous symmetry. A continuum theory proposed by Toner and Tu [3] relates the Vicsek model to the XY model universality class.

Recently, Solon *et al.* [4] studied a discrete model - called active Ising model (AIM) - on a two-dimensional lattice with active particles hopping with bias to the left or to the right and aligning locally like ferromagnetic Ising spins. The flocking transition of this model is very similar to the VM and allows various analytical results. We have studied a generalization for the q-state active Potts model [5, 6] in which self-propelled particles have q internal states (directions of motion) and align locally like Potts spins. In this poster we present the main results obtained for $q = 4$ states.

The microscopic model for $q = 4$ internal states

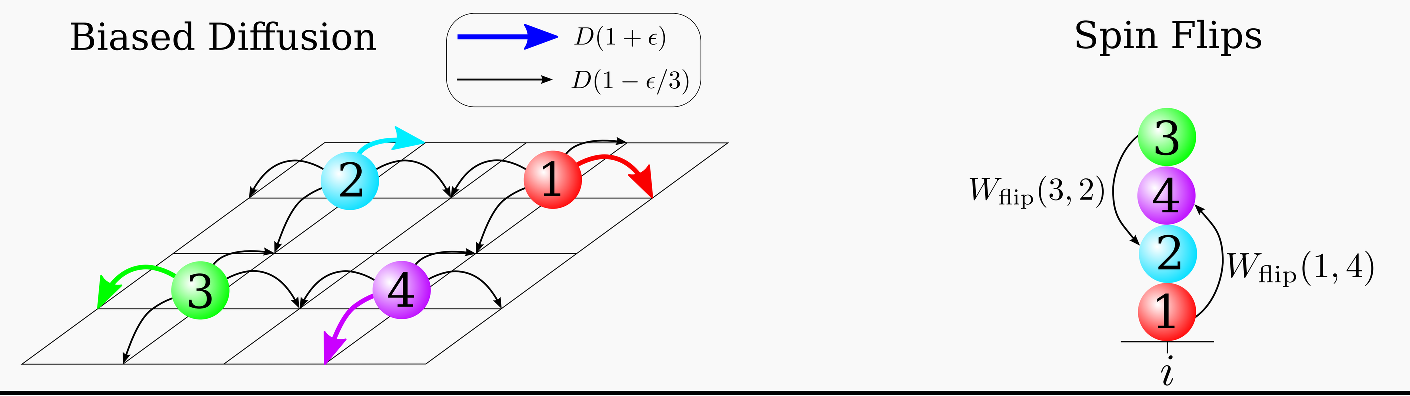
- N particles on a periodic square lattice with L^2 sites. Average density: $\rho_0 = N/L^2$.
- Spin-state of the kth particle on site i: $\sigma_i^k \in \{1, \dots, q\}$.
- Number of particles in state σ on site i: n_i^σ and local density on site i: $\rho_i = \sum_{\sigma=1}^{q} n_i^\sigma$ (no restriction).
- Local magnetisation in state σ on site i: $m_i^\sigma = (qn_i^\sigma - \rho_i)/(q-1)$ and local Hamiltonian on site i:

$$H_i = -\frac{J}{2\rho_i}\sum_{k=1}^{\rho_i}\sum_{l\neq k}(q\delta_{\sigma_i^k,\sigma_i^l} - 1).$$

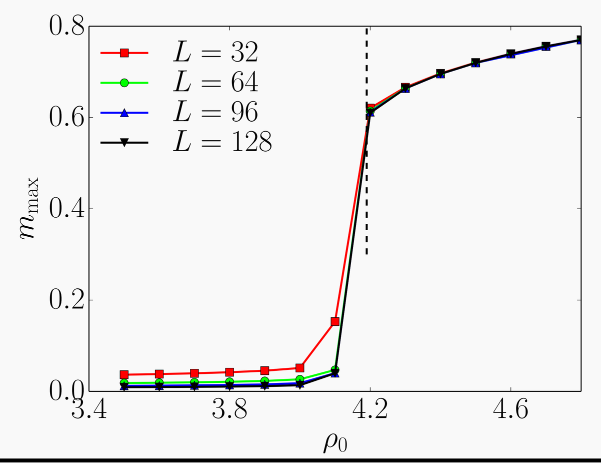
- Flipping rate on site i for a temperature $T = \beta^{-1}$:

$$W_{\rm flip}(\sigma,\sigma') \propto \exp(-\beta\Delta H_i) = \exp\left[-\frac{q\beta J}{\rho_i}(n_i^\sigma - n_i^{\sigma'} - 1)\right].$$

- Hopping rate in preferred direction: $D(1+\epsilon)$ and in other directions: $D[1-\epsilon/(q-1)]$.



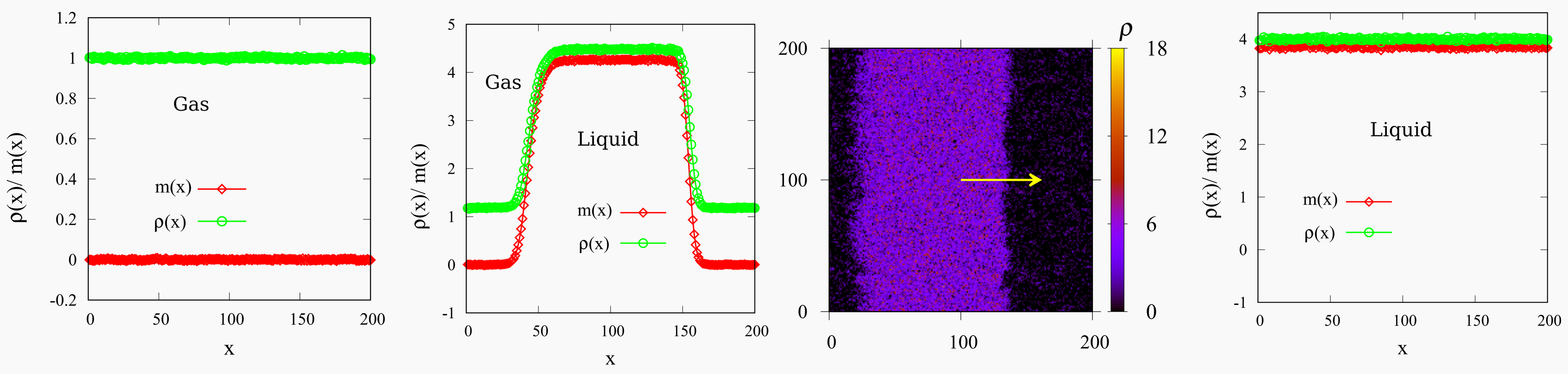
Order-disorder transition ($\epsilon = 0$)



- No phase-separated profiles observed.
- The maximal magnetization $m_{\rm max} = \max_{\{\sigma\}}\left\langle\frac{m_i^\sigma}{\rho_i}\right\rangle$ is discontinuous.
- First order phase transition at the critical point $\rho_0 = \rho_*$ with a spontaneous breaking of the discrete symmetry Z_4:
  - $\rho_0 < \rho_*$: disordered phase ($m_{\rm max} = 0$).
  - $\rho_0 > \rho_*$: ordered phase ($m_{\rm max} > 0$).

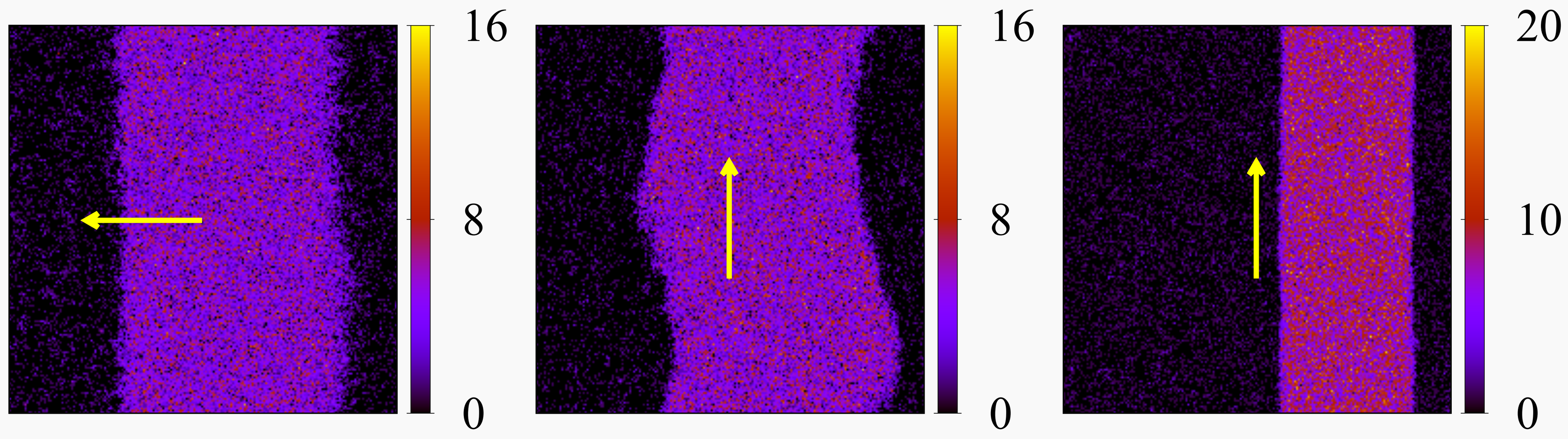
Microscopic simulations: stationary profiles and phase diagrams

- Different phases observed ($\epsilon = 0.9$) via density $\rho(x)$ and magnetisation $m(x)$ profiles:



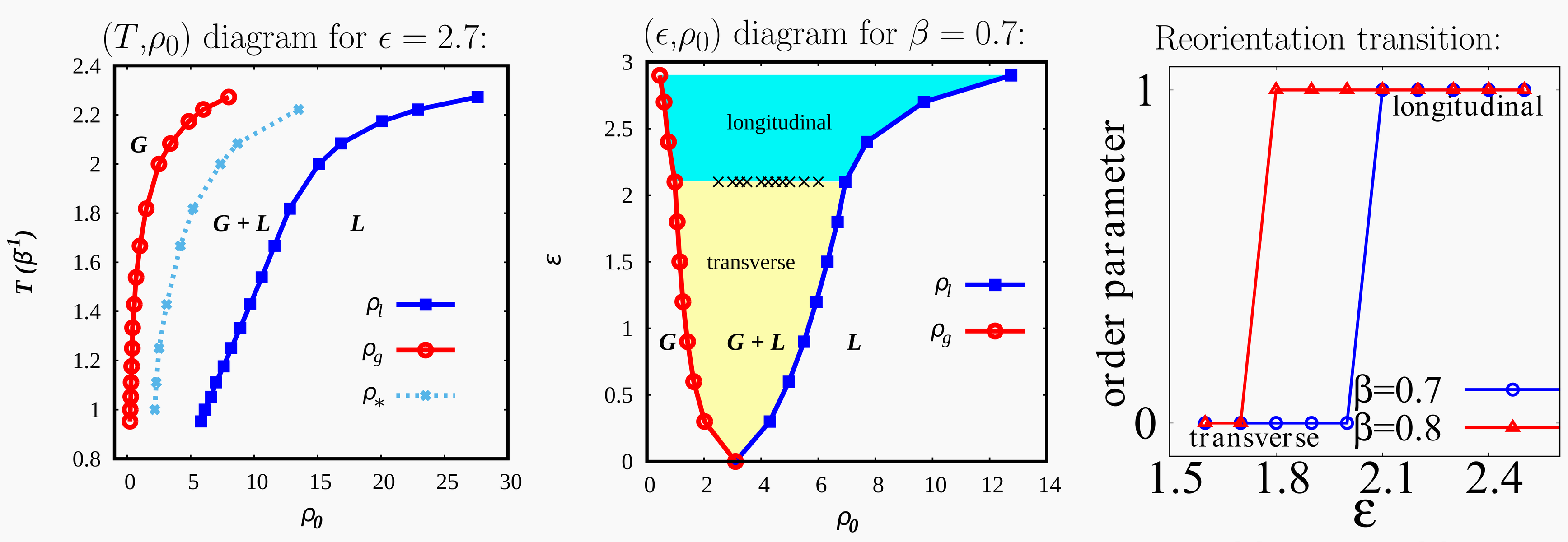
Gas ($\beta = 0.7$, $\rho_0 = 1$) Phase-separation ($\beta = 0.8$, $\rho_0 = 3$) Liquid ($\beta = 0.9$, $\rho_0 = 4$)

- Transverse band motion (small velocities) and longitudinal lane formation (large velocities) :



Transverse band ($\epsilon = 0.9$) Longitudinal lane ($\epsilon = 1.8$) Longitudinal lane ($\epsilon = 2.7$)

- Phase diagrams: the binodals ρ_g and ρ_l delimit the existence of phase-separated profiles.



- The flocking transition is a first-order liquid-gas phase transition, similar to VM [2] and AIM [4], with the absence of the supercritical region. Here the critical temperature is $T_c \simeq 2.4$.
- Reorientation order parameter: $\Delta\theta = \langle\cos(\theta - \phi_i)\rangle$ with θ the direction of the stripe. The transition velocity $\epsilon_*(T)$ is an increasing function of temperature.

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From microscopic model to continuum equations

- Master equations for the microscopic model:

$$\partial_t\langle n_i^\sigma\rangle = D\left(1-\frac{\epsilon}{3}\right)\sum_{p=1}^{4}\left[\langle n_{i-p}^\sigma\rangle - \langle n_i^\sigma\rangle\right] + \frac{4D\epsilon}{3}\left[\langle n_{i-\sigma}^\sigma\rangle - \langle n_i^\sigma\rangle\right] + \sum_{\sigma'\neq\sigma}\left\langle n_i^{\sigma'}W_{\rm flip}(\sigma',\sigma) - n_i^\sigma W_{\rm flip}(\sigma,\sigma')\right\rangle.$$

- Hydrodynamic equations for the density of particles $\rho_\sigma(\mathbf{x},t) = \langle n_i^\sigma(t)\rangle$:

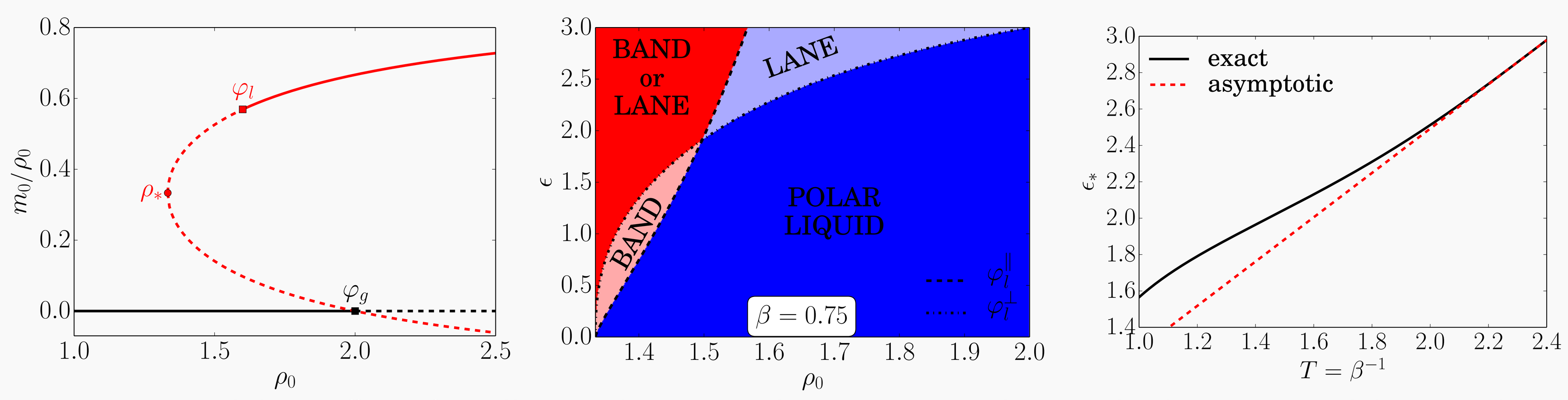
$$\partial_t\rho_\sigma = D_\parallel\partial_\parallel^2\rho_\sigma + D_\perp\partial_\perp^2\rho_\sigma - v\partial_\parallel\rho_\sigma + \sum_{\sigma'\neq\sigma}\left[\frac{4\beta J}{\rho}(\rho_\sigma + \rho_{\sigma'}) - 1 - \frac{r}{\rho} - \alpha\frac{(\rho_\sigma - \rho_{\sigma'})^2}{\rho^2}\right](\rho_\sigma - \rho_{\sigma'}).$$

- $D_\parallel = D(1+\epsilon/3)$ diffusivity in the parallel direction $\mathbf{e}_\parallel = (\cos\phi, \sin\phi)$ and $\partial_\parallel = \mathbf{e}_\parallel\cdot\nabla$.
- $D_\perp = D(1-\epsilon/3)$ diffusivity in the perpendicular direction $\mathbf{e}_\perp = (-\sin\phi, \cos\phi)$ and $\partial_\perp = \mathbf{e}_\perp\cdot\nabla$.
- $v = 4D\epsilon/3$ self-propulsion velocity in the parallel direction. $\phi = \pi(\sigma-1)/2$ favoured direction angle.
- $\alpha = 8(\beta J)^2(1 - 2\beta J/3)$ and r depending on microscopic properties.
- $r = 0$: mean-field equations producing only homogeneous stationary profiles.

Homogeneous solutions and linear stability

- Disordered gas: $\rho_\sigma = \rho_0/4$ for a magnetisation $m_0 = 0$.
- Ordered (polar) liquid: $\rho_\sigma = (\rho_0 + 3m_0)/4$ and $\rho_{\sigma'\neq\sigma} = (\rho_0 - m_0)/4$ for a magnetisation

$$\frac{m_0}{\rho_0} = \frac{\beta J}{\alpha}\left\{1 \pm \sqrt{1 + \frac{(\beta - 1 - r/\rho_0)\alpha}{(\beta J)^2}}\right\} \quad \text{when} \quad \begin{cases}\rho_0 > \rho_* = \frac{8(1-2\beta J/3)r}{1+8(2\beta J-1)(1-2\beta J/3)}\\ T < T_c = (1-\sqrt{22}/8)^{-1} \simeq 2.417\end{cases}$$

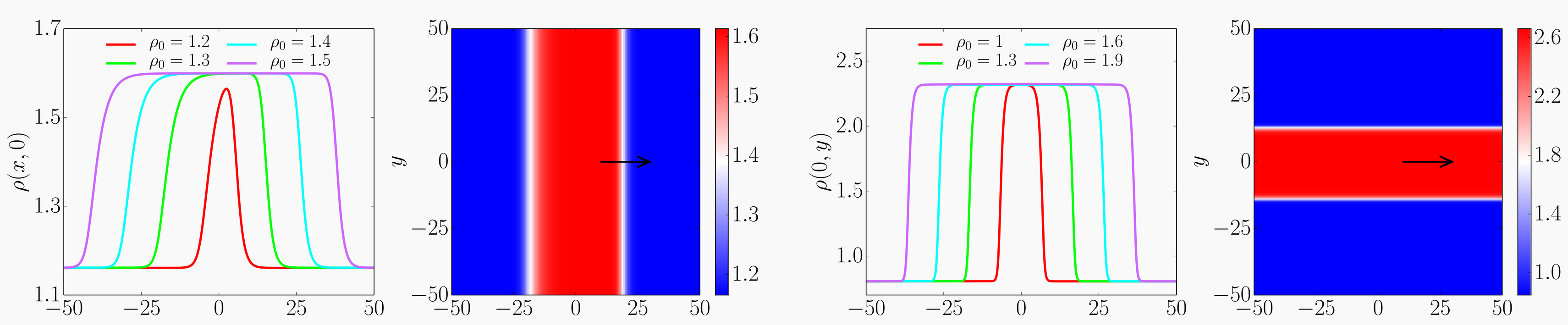


- Linear stability: effect of small perturbations on homogeneous solutions, leading to spinodals φ_g and φ_l:
  - Gas phase stable for $\rho_0 < \varphi_g = r/(2\beta J - 1)$.
  - Transverse bands stable for $\rho_* < \rho_0 < \varphi_l^\perp$ and longitudinal lanes stable for $\rho_* < \rho_0 < \varphi_l^\parallel$.
  - Polar liquid phase stable for $\rho_0 > \varphi_l = \max(\varphi_l^\perp, \varphi_l^\parallel)$, with the highest magnetisation.
  - When $\epsilon = 0$, $\varphi_l = \rho_*$ denotes a first order transition between disordered and ordered phases.
- The reorientation transition occurs when $\varphi_l^\perp = \varphi_l^\parallel$. For $T \to T_c$, the transition velocity is

$$\epsilon_* \simeq 3\left[1 - 0.981\frac{T_c - T}{T_c} + \cdots\right].$$

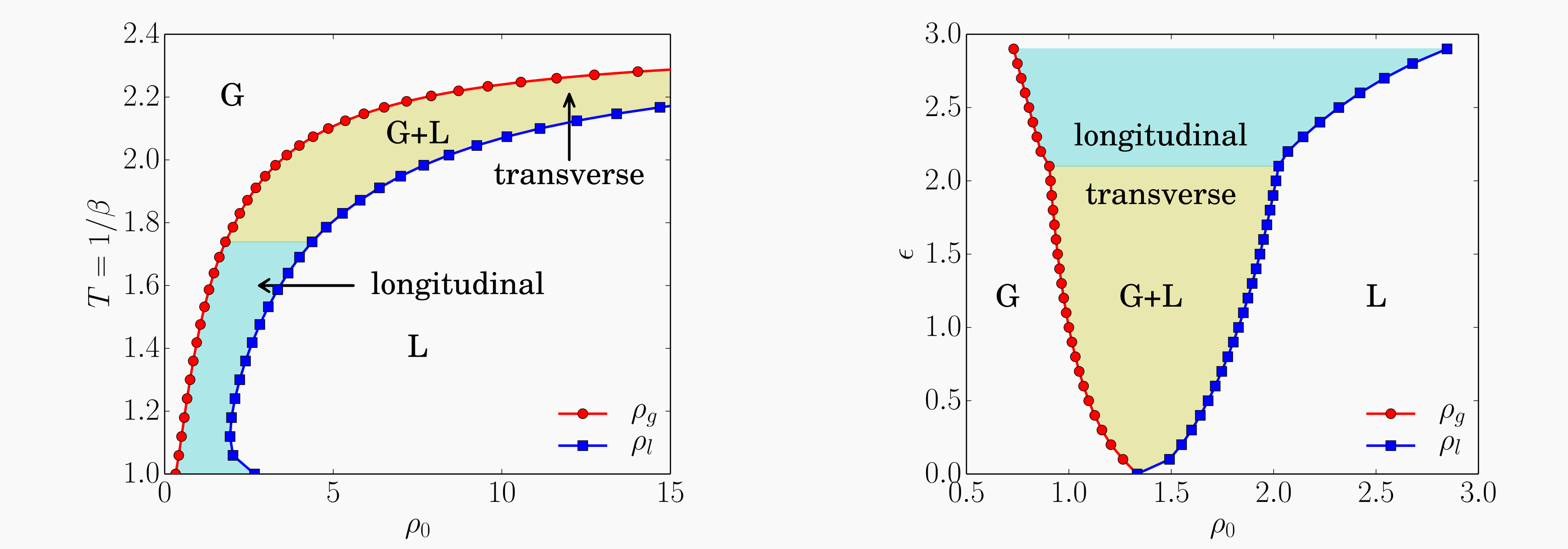
Numerical solutions: density profiles and phase diagrams

- The numerical solutions are obtained with FreeFem++, using the finite element method.
- Transverse band motions and longitudinal lane formations are observed ($\beta = 0.75$):



Transverse band ($\epsilon = 0.3$) Longitudinal lane ($\epsilon = 2.5$)

- First-order liquid-gas phase transition with critical temperature $T_c \simeq 2.417$ and reorientation transition.



Temperature-density diagram ($\epsilon = 2.5$) Velocity-density diagram ($\beta = 0.75$)

Perspectives

- Investigate the $q \to \infty$ limit of the APM and find a model which reproduce the VM in this limit.
- Introduce a restriction on the maximum number of particles allowed on a single site.

References

[1] T. Vicsek, A. Czirok, E. Ben-Jacob, I. Cohen and O. Shochet, Phys. Rev. Lett. **75**, 1226 (1995).

[2] A. P. Solon, H. Chaté and J. Tailleur, Phys. Rev. Lett. **114**, 068101 (2015).

[3] J. Toner and Y. Tu, Phys. Rev. Lett. **75**, 4326 (1995).

[4] A. P. Solon and J. Tailleur, Phys. Rev. Lett. **111**, 078101 (2013); Phys. Rev. E **92**, 042119 (2015).

[5] S. Chatterjee, M. Mangeat, R. Paul and H. Rieger, EPL **130**, 66001 (2020).

[6] M. Mangeat, S. Chatterjee, R. Paul and H. Rieger, Phys. Rev. E **102**, 042601 (2020).

mangeat@lusi.uni-sb.de